

SQUARES OF POSITIVE (p, p) -FORMS

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ABSTRACT. We show that if α is a positive $(2, 2)$ -form then so is α^2 . We also prove that this is no longer true for forms of higher degree.

INTRODUCTION

Recall that a (p, p) -form α in \mathbb{C}^n is called *positive* (we write $\alpha \geq 0$) if for $(1, 0)$ -forms $\gamma_1, \dots, \gamma_{n-p}$ one has

$$\alpha \wedge i\gamma_1 \wedge \bar{\gamma}_1 \wedge \dots \wedge i\gamma_{n-p} \wedge \bar{\gamma}_{n-p} \geq 0.$$

This is a natural geometric condition, positive (p, p) -forms are for example characterized by the following property: for every p -dimensional subspace V and a test function $\varphi \geq 0$ one has

$$\int_V \varphi \alpha \geq 0.$$

It is well known that positive forms are real (that is $\bar{\alpha} = \alpha$) and if β is a $(1, 1)$ -form then

$$(1) \quad \alpha \geq 0, \beta \geq 0 \Rightarrow \alpha \wedge \beta \geq 0.$$

It was shown by Harvey and Knapp [5] (and independently by Bedford and Taylor [1]) that (1) does not hold for all (p, p) and (q, q) -forms α and β , respectively. We refer to Demailly's book [2], pp. 129-132, for a nice and simple introduction to positive forms.

Dinew [3] gave an explicit example of $(2, 2)$ -forms α, β in \mathbb{C}^4 such that $\alpha \geq 0$, $\beta \geq 0$ but $\alpha \wedge \beta < 0$. We will recall it in the next section. The aim of this note is to show the following, somewhat surprising result:

Theorem 1. *Assume that α is a positive $(2, 2)$ -form. Then α^2 is also positive.*

It turns out that this phenomenon holds only for $(2, 2)$ -forms:

Theorem 2. *For every $p \geq 3$ there exists a (p, p) -form α in \mathbb{C}^{2p} such that $\alpha \geq 0$ but $\alpha^2 < 0$.*

We do not know if similar results hold for higher powers of positive forms.

The paper is organized as follows: in Section 1 we present Dinew's criterion for positivity of $(2, 2)$ -forms in \mathbb{C}^4 which reduces the problem to a certain property of 6×6 matrices. Further simplification reduces the problem to 4×4 matrices. We then solve it in Section 2.

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This is the most technical part of the paper. Higher degree forms are analyzed in Section 3 where a counterpart of Dinew's criterion is showed and Theorem 2 is proved.

1. DINEW'S CRITERION

Without loss of generality we may assume that $n = 4$. Let $\omega_1, \dots, \omega_4$ be a basis of $(\mathbb{C}^4)^*$ such that

$$dV := i\omega_1 \wedge \bar{\omega}_1 \wedge \dots \wedge i\omega_4 \wedge \bar{\omega}_4 = \omega_1 \wedge \dots \wedge \omega_4 \wedge \bar{\omega}_1 \wedge \dots \wedge \bar{\omega}_4 > 0.$$

Set

$$\begin{aligned} \Omega_1 &:= \omega_1 \wedge \omega_2, & \Omega_2 &:= \omega_1 \wedge \omega_3, & \Omega_3 &:= \omega_1 \wedge \omega_4, \\ \Omega_4 &:= \omega_2 \wedge \omega_3, & \Omega_5 &:= -\omega_2 \wedge \omega_4, & \Omega_6 &:= \omega_3 \wedge \omega_4. \end{aligned}$$

Then

$$\Omega_j \wedge \Omega_k = \begin{cases} \omega_1 \wedge \dots \wedge \omega_4, & \text{if } k = 7 - j \\ 0, & \text{otherwise.} \end{cases}$$

Every $(2, 2)$ -form α we can associate with a 6×6 -matrix $A = (a_{jk})$ by

$$\alpha = \sum_{j,k} a_{jk} \Omega_j \wedge \bar{\Omega}_k.$$

For

$$\beta = \sum_{j,k} b_{jk} \Omega_j \wedge \bar{\Omega}_k$$

we have

$$(2) \quad \alpha \wedge \beta = \sum_{j,k} a_{jk} b_{7-j, 7-k} dV.$$

The key will be the following criterion from [3]:

Theorem 3. $\alpha \geq 0$ iff $\bar{z}Az^T \geq 0$ for all $z \in \mathbb{C}^6$ with $z_1z_6 + z_2z_5 + z_3z_4 = 0$.

Sketch of proof. For $\gamma_1 = b_1\omega_1 + \dots + b_4\omega_4$, $\gamma_2 = c_1\omega_1 + \dots + c_4\omega_4$ we have

$$\begin{aligned} i\gamma_1 \wedge \bar{\gamma}_1 \wedge i\gamma_2 \wedge \bar{\gamma}_2 &= \sum_{j,k,l,m=1}^4 b_j \bar{b}_k c_l \bar{c}_m \omega_j \wedge \omega_l \wedge \bar{\omega}_k \wedge \bar{\omega}_m \\ &= \sum_{\substack{j < l \\ k < m}} (b_j c_l - b_l c_j) \overline{(b_k c_m - b_m c_k)} \omega_j \wedge \omega_l \wedge \bar{\omega}_k \wedge \bar{\omega}_m. \end{aligned}$$

It is now enough to show that the image of the mapping

$$\mathbb{C}^8 \ni (b_1, \dots, b_4, c_1, \dots, c_4) \mapsto$$

$$(b_1c_2 - b_2c_1, b_1c_3 - b_3c_1, b_1c_4 - b_4c_1, b_2c_3 - b_3c_2, -b_2c_4 + b_4c_2, b_3c_4 - b_4c_3) \in \mathbb{C}^6$$

is precisely $\{z \in \mathbb{C}^6 : z_1z_6 + z_2z_5 + z_3z_4 = 0\}$. □

Using Theorem 3 and an idea from [3] we can show:

Proposition 4. *The form*

$$\alpha_a = \sum_{j=1}^6 \Omega_j \wedge \bar{\Omega}_j + a \Omega_1 \wedge \bar{\Omega}_6 + \bar{a} \Omega_6 \wedge \bar{\Omega}_1$$

is positive if and only if $|a| \leq 2$.

Proof. We have

$$\bar{z}Az^T = |z|^2 + 2\operatorname{Re}(a\bar{z}_1z_6) \geq 2|z_1z_6| + 2|z_2z_5 + z_3z_4| + 2\operatorname{Re}(a\bar{z}_1z_6).$$

If $z_1z_6 + z_2z_5 + z_3z_4 = 0$ and $|a| \leq 2$ we clearly get $\bar{z}Az^T \geq 0$. If we take z_1, z_6 with $\bar{z}_1z_6 = -\bar{a}$, $|z_1| = |z_6|$ and z_2, \dots, z_5 with $z_2z_5 + z_3z_4 = -z_1z_6$ then $\bar{z}Az^T = 2|a|(2 - |a|)$. \square

By (2)

$$\alpha_a \wedge \alpha_b = 2(3 + \operatorname{Re}(a\bar{b}))dV.$$

Therefore α_2, α_{-2} are real and positive but $\alpha_2 \wedge \alpha_{-2} < 0$.

In view of Theorem 3 we see that Theorem 1 is equivalent to the following:

Theorem 5. *Let $A = (a_{jk}) \in \mathbb{C}^{6 \times 6}$ be hermitian and such that $\bar{z}Az^T \geq 0$ for $z \in \mathbb{C}^6$ with $z_1z_6 + z_2z_5 + z_3z_4 = 0$. Then*

$$\sum_{j,k=1}^6 a_{jk} a_{7-j,7-k} \geq 0.$$

We will need the following technical reduction:

Lemma 6. *For every $(2, 2)$ -form α in \mathbb{C}^4 we can find a basis $\omega_1, \dots, \omega_4$ of $(\mathbb{C}^4)^*$ such that*

$$(3) \quad \alpha \wedge \omega_1 \wedge \omega_2 \wedge \bar{\omega}_1 \wedge \bar{\omega}_j = \alpha \wedge \omega_1 \wedge \omega_2 \wedge \bar{\omega}_2 \wedge \bar{\omega}_j = 0$$

for $j = 3, 4$.

Proof. We may assume that $\alpha \neq 0$, then we can find $\omega_1, \omega_2 \in (\mathbb{C}^4)^*$ such that

$$(4) \quad \alpha \wedge \omega_1 \wedge \omega_2 \wedge \bar{\omega}_1 \wedge \bar{\omega}_2 = \alpha \wedge i\omega_1 \wedge \bar{\omega}_1 \wedge i\omega_2 \wedge \bar{\omega}_2 \neq 0.$$

By V_1 denote the subspace spanned by ω_1, ω_2 and by V_2 the subspace of all $\omega \in (\mathbb{C}^4)^*$ satisfying (3) with ω_j replaced by ω . Then $\dim V_1 = 2$, $\dim V_2 \geq 2$ and by (4) we infer $V_1 \cap V_2 = \{0\}$, hence $(\mathbb{C}^4)^* = V_1 \oplus V_2$. \square

2. PROOF OF THEOREM 5

By Lemma 6 we may assume that the matrix from Theorem 5 satisfies

$$a_{26} = a_{36} = a_{46} = a_{56} = 0$$

and

$$a_{62} = a_{63} = a_{64} = a_{65} = 0.$$

Then

$$\sum_{j,k=1}^6 a_{jk} a_{7-j,7-k} = \sum_{j,k=2}^5 a_{jk} a_{7-j,7-k} + 2(a_{11}a_{66} + |a_{16}|^2).$$

Therefore Theorem 5 is in fact equivalent to the following result:

Theorem 7. *Let $A = (a_{jk}) \in \mathbb{C}^{4 \times 4}$ be hermitian and such that $\bar{z}Az^T \geq 0$ for $z \in \mathbb{C}^4$ with $z_1z_4 + z_2z_3 = 0$. Then*

$$(5) \quad \sum_{j,k=1}^4 a_{jk} a_{5-j,5-k} \geq 0.$$

Proof. Write

$$A = \begin{pmatrix} \lambda_1 & a & b & \alpha \\ \bar{a} & \lambda_2 & \beta & -c \\ \bar{b} & \bar{\beta} & \lambda_3 & -d \\ \bar{\alpha} & -\bar{c} & -\bar{d} & \lambda_4 \end{pmatrix}.$$

It satisfies the assumption of the theorem if and only if for every $z \in \mathbb{C}^4$ of the form $z = (1, \zeta, w, -\zeta w)$ one has $\bar{z}Az^T \geq 0$. We can then compute

$$\begin{aligned} \bar{z}Az^T = & \lambda_1 + 2\operatorname{Re}(a\zeta) + \lambda_2|\zeta|^2 \\ & + 2\operatorname{Re}[(b - \alpha\zeta + \beta\bar{\zeta} + c|\zeta|^2)w] \\ & + (\lambda_3 + 2\operatorname{Re}(d\zeta) + \lambda_4|\zeta|^2)|w|^2. \end{aligned}$$

Therefore A satisfies the assumption iff $\lambda_j \geq 0$,

$$(6) \quad |a| \leq \sqrt{\lambda_1\lambda_2}, \quad |b| \leq \sqrt{\lambda_1\lambda_3}, \quad |c| \leq \sqrt{\lambda_2\lambda_4}, \quad |d| \leq \sqrt{\lambda_3\lambda_4},$$

and for every $\zeta \in \mathbb{C}$

$$(7) \quad |b - \alpha\zeta + \beta\bar{\zeta} + c|\zeta|^2|^2 \leq (\lambda_1 + 2\operatorname{Re}(a\zeta) + \lambda_2|\zeta|^2) (\lambda_3 + 2\operatorname{Re}(d\zeta) + \lambda_4|\zeta|^2).$$

In our case (5) is equivalent to

$$4\operatorname{Re}(a\bar{d} + b\bar{c}) \leq 2(\lambda_1\lambda_4 + \lambda_2\lambda_3) + 2(|\alpha|^2 + |\beta|^2).$$

We will in fact prove something more:

$$(8) \quad 4\operatorname{Re}(a\bar{d} + b\bar{c}) \leq (\sqrt{\lambda_1\lambda_4} + \sqrt{\lambda_2\lambda_3})^2 + (|\alpha| + |\beta|)^2.$$

Without loss of generality we may assume that

$$\operatorname{Re}(a\bar{d}) > 0, \quad \operatorname{Re}(b\bar{c}) > 0,$$

for if for example $\operatorname{Re}(a\bar{d}) \leq 0$ then by (6)

$$4\operatorname{Re}(a\bar{d} + b\bar{c}) \leq 4\operatorname{Re}(b\bar{c}) \leq 4\sqrt{\lambda_1\lambda_2\lambda_3\lambda_4} \leq (\sqrt{\lambda_1\lambda_4} + \sqrt{\lambda_2\lambda_3})^2.$$

Set $u := \operatorname{Re}(a\bar{d})$ and $\zeta := -r\bar{d}/|d|$, where $r > 0$ will be determined later. Then the right-hand side of (7) we can write as follows

$$\begin{aligned} & \left(\lambda_1 - \frac{2ur}{|d|} + \lambda_2 r^2\right) (\lambda_3 - 2r|d| + \lambda_4 r^2) \\ &= (\lambda_1 + \lambda_2 r^2) (\lambda_3 + \lambda_4 r^2) + 4ur^2 - 2r \left[\lambda_1 |d| + \lambda_3 \frac{u}{|d|} + r^2 \left(\lambda_2 |d| + \lambda_4 \frac{u}{|d|} \right) \right] \\ &\leq (\lambda_1 + \lambda_2 r^2) (\lambda_3 + \lambda_4 r^2) + 4ur^2 - 4r^2 (\sqrt{\lambda_1 \lambda_4} + \sqrt{\lambda_2 \lambda_3}) \sqrt{u} \\ &= (\sqrt{\lambda_1 \lambda_4} + \sqrt{\lambda_2 \lambda_3} - 2\sqrt{u})^2 r^2 + (\sqrt{\lambda_1 \lambda_3} - \sqrt{\lambda_2 \lambda_4} r^2)^2. \end{aligned}$$

For $r = (\frac{\lambda_1 \lambda_3}{\lambda_2 \lambda_4})^{1/4}$ from (7) we thus obtain

$$\left| \frac{b}{r} + \frac{\bar{d}}{|d|} \alpha - \frac{d}{|d|} \beta + cr \right| \leq \sqrt{\lambda_1 \lambda_4} + \sqrt{\lambda_2 \lambda_3} - 2\sqrt{u}.$$

We also have

$$\left| \frac{b}{r} + \frac{\bar{d}}{|d|} (\alpha - \bar{\beta}) + cr \right| \geq \left| \frac{b}{r} + cr \right| - |\alpha| - |\beta| \geq 2\sqrt{\operatorname{Re}(b\bar{c})} - |\alpha| - |\beta|$$

and therefore

$$2\sqrt{\operatorname{Re}(a\bar{d})} + 2\sqrt{\operatorname{Re}(b\bar{c})} \leq \sqrt{\lambda_1 \lambda_4} + \sqrt{\lambda_2 \lambda_3} + |\alpha| + |\beta|.$$

To get (8) we can now use the following fact: if $0 \leq a_1 \leq x$, $0 \leq a_2 \leq x$ and $a_1 + a_2 \leq x + y$ then $a_1^2 + a_2^2 \leq x^2 + y^2$. This can be easily verified: if $a_1 + a_2 \leq x$ then $a_1^2 + a_2^2 \leq x^2$ and if $a_1 + a_2 \geq x$ then

$$x^2 + y^2 \geq x^2 + (a_1 + a_2 - x)^2 = a_1^2 + a_2^2 + 2x(x - a_1)(x - a_2).$$

□

3. (p, p) -FORMS IN \mathbb{C}^{2p}

In \mathbb{C}^{2p} we choose the positive volume form

$$dV := idz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge idz_{2p} \wedge d\bar{z}_{2p} = dz_1 \wedge \cdots \wedge dz_{2p} \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{2p}.$$

By \mathcal{I} we will denote the set of subscripts $J = (j_1, \dots, j_p)$ such that $1 \leq j_1 < \cdots < j_p \leq 2p$. For every $J \in \mathcal{I}$ there exists unique $J' \in \mathcal{I}$ such that $J \cup J' = \{1, \dots, 2p\}$. We also denote $dz_J = dz_{j_1} \wedge \cdots \wedge dz_{j_p}$ and $\varepsilon_J = \pm 1$ is defined in such a way that

$$dz_J \wedge dz_{J'} = \varepsilon_J dz_1 \wedge \cdots \wedge dz_{2p}.$$

Note that

$$(9) \quad \varepsilon_J \varepsilon_{J'} = (-1)^p.$$

Every (p, p) -form α in \mathbb{C}^{2p} we can associate with a $N \times N$ -matrix (a_{JK}) , where

$$N = \#\mathcal{I} = \frac{(2p)!}{(p!)^2},$$

by

$$(10) \quad \alpha = \sum_{J,K} a_{JK} idz_{j_1} \wedge d\bar{z}_{k_1} \wedge \cdots \wedge idz_{j_p} \wedge d\bar{z}_{k_p} = i^{p^2} \sum_{J,K} a_{JK} dz_J \wedge d\bar{z}_K$$

(note that $(-1)^{p(p-1)/2} i^p = i^{p^2}$). Then

$$(11) \quad \alpha^2 = \sum_{J,K} \varepsilon_J \varepsilon_K a_{JK} a_{J'K'} dV$$

and for $\gamma_1, \dots, \gamma_p \in (\mathbb{C}^{2p})^*$

$$\alpha \wedge i\gamma_1 \wedge \bar{\gamma}_1 \wedge \cdots \wedge i\gamma_p \wedge \bar{\gamma}_p = \sum_{J,K} a_{JK} \gamma_1 \wedge \cdots \wedge \gamma_p \wedge dz_J \wedge \overline{(\gamma_1 \wedge \cdots \wedge \gamma_p \wedge dz_K)}.$$

Therefore (a_{JK}) has to be positive semi-definite on the image of Plücker embedding

$$(12) \quad ((\mathbb{C}^{2p})^*)^p \ni (\gamma_1, \dots, \gamma_p) \mapsto \left(\frac{\gamma_1 \wedge \cdots \wedge \gamma_p \wedge dz_J}{dz_1 \wedge \cdots \wedge dz_{2p}} \right)_{J \in \mathcal{I}} \in \mathbb{C}^N$$

which is well known to be a variety in \mathbb{C}^N (see e.g. [4], p. 64).

We are now ready to prove Theorem 2:

Proof of Theorem 2. First note that it is enough to show it for $p = 3$. For if α is a $(3,3)$ -form in \mathbb{C}^6 such that $\alpha \geq 0$ and $\alpha^2 < 0$ then for $p > 3$ we set

$$\beta := idz_7 \wedge d\bar{z}_7 + \cdots + idz_{2p} \wedge d\bar{z}_{2p}.$$

We now have $\alpha \wedge \beta^{p-3} \geq 0$ but $(\alpha \wedge \beta^{p-3})^2 = \alpha^2 \wedge \beta^{2p-6} < 0$.

Set $p = 3$, so that $N = 20$, and order $\mathcal{I} = \{J_1, \dots, J_{20}\}$ lexicographically. Then the image of Plücker embedding is in particular contained in the quadric

$$(13) \quad z_1 z_{20} - z_{10} z_{11} + z_5 z_{16} - z_2 z_{19} = 0.$$

For positive a, λ, μ to be determined later define

$$\alpha := \lambda i(dz_{J_1} \wedge d\bar{z}_{J_1} + dz_{J_{20}} \wedge d\bar{z}_{J_{20}}) + \mu i \sum_{\substack{k \in \{2,5,10, \\ 11,16,19\}}} dz_{J_k} \wedge d\bar{z}_{J_k} + a(dz_{J_1} \wedge d\bar{z}_{J_{20}} + dz_{J_{20}} \wedge d\bar{z}_{J_1}).$$

Then, similarly as in the proof of Proposition 4,

$$\begin{aligned} \bar{z} A z^T &= \lambda(|z_1|^2 + |z_{20}|^2) + \mu(|z_2|^2 + |z_5|^2 + |z_{10}|^2 + |z_{11}|^2 + |z_{16}|^2 + |z_{19}|^2) + 2a \operatorname{Re}(\bar{z}_1 z_{20}) \\ &\geq 2(\lambda - a)|z_1 z_{20}| + 2\mu| -z_{10} z_{11} + z_5 z_{16} - z_2 z_{19}| \\ &= 2(\lambda + \mu - a)|z_1 z_{20}| \end{aligned}$$

if z satisfies (13). Therefore $\alpha \geq 0$ if $a \leq \lambda + \mu$.

On the other hand by (11) and (9)

$$\alpha^2 = 2(\lambda^2 + 3\mu^2 - a^2)dV.$$

We see that if we take $a = \lambda + \mu$ and $\lambda > \mu > 0$ then $\alpha \geq 0$ but $\alpha^2 < 0$. □

REFERENCES

- [1] E. BEDFORD, B.A. TAYLOR, *Simple and positive vectors in the exterior algebra of \mathbb{C}^n* , preprint, 1974
- [2] J.-P. DEMAILLY, *Complex Analytic and Differential Geometry*, monograph, 1997, available at <http://www-fourier.ujf-grenoble.fr/~demailly>
- [3] S. DINEW, *On positive $\mathbb{C}_{(2,2)}(\mathbb{C}^4)$ forms*, preprint, 2006
- [4] J. HARRIS, *Algebraic Geometry. A First Course*, Graduate Texts in Math. 133, Springer, 1995
- [5] R. HARVEY, A.W. KNAPP, *Positive (p, p) forms, Wirtinger's inequality, and currents*, Value Distribution Theory, R.O. Kujala and A.L. Vitter III (ed.), Part A, pp. 43–62, Dekker, 1974

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